

Characteristic properties of the scattering data for the mKdV equation on the half-line

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Abstract

In this paper we describe characteristic properties of the scattering data of the compatible eigenvalue problem for the pair of differential equations related to the modified Korteweg-de Vries (mKdV) equation whose solution is defined in some half-strip $(0 < x < \infty) \times [0, T]$, or in the quarter plane $(0 < x < \infty) \times (0 < t < \infty)$. We suppose that this solution has a C^∞ initial function vanishing as $x \rightarrow \infty$, and C^∞ boundary values, vanishing as $t \rightarrow \infty$ when $T = \infty$. We study the corresponding scattering problem for the compatible Zakharov-Shabat system of differential equations associated with the mKdV equation and obtain a representation of the solution of the mKdV equation through Marchenko integral equations of the inverse scattering method. The kernel of these equations is valid only for $x \geq 0$ and it takes into account all specific properties of the pair of compatible differential equations in the chosen half-strip or in the quarter plane. The main result of the paper is the collection A–B–C of characteristic properties of the scattering functions given below.

1 Introduction

1.1

Initial value problems on the whole line for nonlinear integrable equations such as the nonlinear Schrödinger equation, the Korteweg-de Vries equation, the sine-Gordon equation, etc. are well studied. The solvability of the Cauchy problem, multi-soliton solutions, the proof that these nonlinear equations are completely integrable infinite-dimensional Hamiltonian systems are the most significant results in the soliton theory on the whole line.

At the same time the initial-boundary value problem on the half-line for nonlinear integrable equations has not been studied so far. In the last decade attention to those problem has strongly increased. Among papers [2]-[15], [20]-[33], [36]-[47] devoted to this problem, the most interesting results were obtained by A.S. Fokas [20], A.S. Fokas and A.R. Its [22]-[24]. Later, in [26] A.S. Fokas has proposed a general method for solving boundary value problems for two-dimensional linear and integrable nonlinear

partial differential equations. This method, which was further developed in [21], [27]-[29], is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann-Hilbert problem in the complex plane of the spectral parameter. The spectral functions determining the Riemann-Hilbert problem are expressed in terms of the initial and boundary values of the solution. The fact that these initial and boundary values are in general related can be expressed in a simple way in terms of a global relation satisfied by the corresponding spectral functions.

In the framework of this approach we recently found characteristic properties of the scattering data for the compatible Zakharov-Shabat eigenvalue problem associated with focusing and defocusing nonlinear Schrödinger equations on the half-line with initial and boundary functions of Schwartz type [9].

Recently in [11] (see also [33]) an initial-boundary value problem for the mKdV equation on the half-line was analyzed by expressing the solution in terms of the solution of a matrix Riemann-Hilbert problem in the complex k -plane. In particular, it is shown that for a subclass of boundary conditions, the “linearizable boundary conditions”, all spectral functions can be computed from the given initial data by using algebraic manipulations of some “global relation”. Thus in this case, the problem on the half-line can be solved as efficiently as the problem on the whole line.

But the general initial-boundary value problem on the half-line remains non-linearizable. Characteristic properties of the spectral functions were not considered in [11]. In this connection the characterization of the spectral functions becomes important. Besides, a description of the characteristic properties of the scattering or spectral data is a very important problem in itself [35].

Most of papers initiated of problems on the half-line deal with nonlinear dynamics of the spectral or scattering data. We prefer the approach of A.S. Fokas and A.R. Its where scattering (spectral) data have trivial dynamics. But then analytic properties of the scattering data are more complicated. Therefore it is necessary to give their complete description when, of course, initial and boundary functions belong to suitable classes of functions. We introduce spectral data in a natural way as in ordinary scattering problems:

1. First, a “scattering matrix” for the x -equation (by initial function).
2. Then, a “scattering matrix” for the t -equation (by boundary functions),
3. Finally, a “scattering matrix” for the compatible x and t -equations as by-product.

In this case a kernel of the Marchenko integral equations or a jump matrix of the corresponding Riemann-Hilbert problem has an explicit x, t dependence. That makes possible to study the asymptotic behavior of the solution of the non-linear problem by using, for example, the powerful steepest descent method of P. Deift and X. Zhou [16, 17] while the non-linear dynamics of the spectral or scattering data makes almost impossible to obtain an effective asymptotics of the solution.

1.2

In this paper we consider the problem to characterize “scattering data” for a compatible pair of differential equations attached to the modified Korteweg-de Vries (mKdV) equation. Let $q(x, t)$ be a real-valued solution of the mKdV equation

$$\begin{aligned} q_t + q_{xxx} - 6\lambda q^2 q_x &= 0, \\ x \in \mathbb{R}_+, \quad t \in [0, T], \quad T \leq \infty, \quad \lambda = \pm 1 \end{aligned} \quad (1.1)$$

in the half-strip or quarter xt -plane and suppose the initial function

$$q(x, 0) = u(x) \text{ with } x \in \mathbb{R}_+;$$

and the boundary values

$$q(0, t) = v(t) \quad q_x(0, t) = v_1(t) \quad q_{xx}(0, t) = v_2(t) \text{ with } t \in [0, T], \quad T \leq \infty$$

are C^∞ , and $u(x) \in \mathcal{S}(\mathbb{R}_+)$, where $\mathcal{S}(\mathbb{R}_+)$ is the Schwartz space of rapidly decreasing functions on \mathbb{R}_+ , i.e. C^∞ functions whose derivatives of any order $n \geq 0$ vanish at infinity faster than any negative power of x . For $T = \infty$ the boundary values are also supposed to be rapidly decreasing: $v(t), v_1(t), v_2(t) \in \mathcal{S}(\mathbb{R}_+)$.

Remark. We consider here the IBV problem for the mKdV equation (1.1) in the first quarter ($x \geq 0, t \geq 0$) of the xt -plane. This problem differs from that studied in [11] which is also on the first quarter but for the mKdV equation of the form:

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0.$$

This form can be easily reduced to (1.1), but then the IBV problem is on the second quarter ($x \leq 0, t \geq 0$) of the xt -plane. Scattering (spectral) data for that problem have different analytic properties. It is well-known that for KdV and mKdV equations there are differences between the IBV problems for $x > 0$ and for $x < 0$.

To study the solution $q(x, t)$ we shall use spectral analysis of a compatible eigenvalue problem for the linear x -equation

$$\begin{aligned} w_x + ik\sigma_3 w &= Q(x, t)w, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ Q(x, t) &= \begin{pmatrix} 0 & q(x, t) \\ \lambda q(x, t) & 0 \end{pmatrix} \end{aligned} \quad (1.2)$$

and for the linear t -equation

$$\begin{aligned} w_t + 4ik^3 \sigma_3 w &= \hat{Q}(x, t, k)w, \\ \hat{Q}(x, t, k) &= 2Q^3(x, t) - Q_{xx} - 2ik(Q^2(x, t) + Q_x(x, t))\sigma_3 + 4k^2 Q(x, t). \end{aligned} \quad (1.3)$$

This is the well-known Ablowitz-Kaup-Newel-Segur [1] or Zakharov-Shabat [48] system of linear equations, which are compatible if and only if $q(x, t)$ satisfies the mKdV equation.

The main goal of the present paper is to study the scattering problem for compatible differential equations (1.2) and (1.3) on the half-strip or on the quarter of the xt -plane. We will combine the Marchenko integral equation and the corresponding Riemann–Hilbert problem in our approach to obtain characteristic properties of scattering data. We get a description of these characteristic properties and obtain a representation of the solution of the mKdV equation through Marchenko integral equations of the inverse scattering method. The kernel of these equations is valid for $x \geq 0$ only and it takes into account all specific properties occurred for compatible differential equations. In particular, the solution of the mKdV equation given by our method does not have continuation for $x < 0$. It is well defined on the half-strip or on the quarter of the xt -plane. Then, if one is using our representation of the solution of the mKdV equation and the explicit (x, t) -dependence of the kernel of the Marchenko integral equations or the jump matrix in the Riemann-Hilbert problem one can easily obtain the asymptotic behavior of the solution in the same way as in [16, 17] or [34].

1.3 Scattering data: definition and properties

Let $q(x, t)$ be a real-valued solution of Equation (1.1) with initial and boundary functions satisfying smoothness and decreasing assumptions described above. Let us define $\Sigma = \{k \in \mathbb{C} \mid \text{Im } k^3 = 0\}$ and domains $\Omega_1, \dots, \Omega_6$ as depicted on Figure 1.

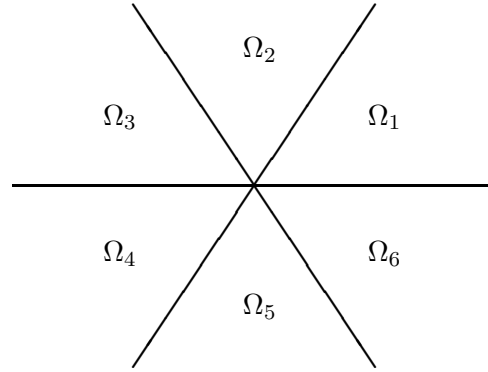


Figure 1: $\Sigma = \{k \in \mathbb{C} \mid \text{Im } k^3 = 0\}$ and $\Omega_1, \dots, \Omega_6$.

Scattering data are introduced as follows.

1. The initial function $u(x) = q(x, 0)$ and the x -equation (1.2) with $t = 0$ define the Jost solution $\Psi(x, 0, k) = \exp(-ikx\sigma_3) + o(1)$, $x \rightarrow \infty$ and a “scattering matrix”

$$S(k) := \Psi^{-1}(0, 0, k) = \begin{pmatrix} s_2^+(k) & -s_1^+(k) \\ -s_2^-(k) & s_1^-(k) \end{pmatrix}, \quad s_1^-(k) = \bar{s}_2^+(\bar{k}), \quad s_2^-(k) = \lambda \bar{s}_1^+(\bar{k}).$$

In particular, they define

- the spectral function $r(k) = -s_2^-(k)/s_2^+(k)$, called the “reflection coefficient”,
- eigenvalues $k_j \in \mathbb{C}_+$, $j = 1, \dots, n$, which are zeros of $s_2^+(k)$,
- numbers $m_j = [is_1^+(k_j)s_2^+(k_j)]^{-1}$, $k_j \in \mathbb{C}_+$.

2. Boundary data $v(t) = q(0, t)$, $v_1(t) = q_x(0, t)$, $v_2(t) = q_{xx}(0, t)$, with $t \in [0, T]$, $T \leq \infty$ and the t -equation with $x = 0$ define a solution $Y(0, t, k) = \exp(-4ik^3 t \sigma_3)$, $t \geq T$, then a “scattering matrix”

$$P(k) := Y(0, 0, k) = \begin{pmatrix} p_1^-(k) & p_1^+(k) \\ p_2^-(k) & p_2^+(k) \end{pmatrix} \quad p_1^-(k) = \bar{p}_2^+(\bar{k}), \quad p_2^-(k) = \lambda \bar{p}_1^+(\bar{k}),$$

and, together with $S(k)$, another “scattering matrix”

$$R(k) = S(k)P(k) = \begin{pmatrix} r_1^-(k) & r_1^+(k) \\ r_2^-(k) & r_2^+(k) \end{pmatrix}.$$

Now we introduce:

- one more spectral function $c(k) = \frac{p_2^-(k)}{s_2^+(k)r_1^-(k)}$, $k \in \Omega_2$,
- eigenvalues $z_j \in \Omega_2$, $j = 1, \dots, m$, which are zeros of $r_1^-(k)$,
- numbers $m_j^2 = -i \operatorname{Res}_{k=z_j} c(k)$ ($z_j \in \Omega_2$), which depend on the initial and boundary functions.

Scattering data. We define the set

$$\mathcal{R} = \{k_1, \dots, k_n \in \mathbb{C}_+; z_1, \dots, z_m \in \Omega_2; r(k), k \in \mathbb{R}; c(k), k \in \Omega_2\}$$

as “scattering data” of the compatible eigenvalue problem for the system of differential equations (1.2)-(1.3) with $q(x, t)$ satisfying the mKdV equation (1.1).

Recovering of $q(x, t)$ from scattering data

Then we prove that the solution $q(x, t)$ of the non-linear problem (1.1) can be written

$$q(x, t) = -2\lambda K_2(x, x, t) \tag{1.4}$$

where $K_2(x, y, t)$, together with $K_1(x, y, t)$, satisfies the Marchenko integral equations:

$$K_1(x, y, t) + \lambda \int_x^\infty K_2(x, z, t) H(z + y, t) dz = 0 \text{ for } 0 \leq x < y < \infty, \tag{1.5}$$

$$K_2(x, y, t) + H(x + y, t) + \int_x^\infty K_1(x, z, t) H(z + y, t) dz = 0 \tag{1.6}$$

with kernel

$$\begin{aligned} H(x, t) = & \frac{1-\lambda}{2} \left(\sum_{k_j \in \Omega_1 \cup \Omega_3} m_j e^{ik_j x + 8ik_j^3 t} + \sum_{z_j \in \Omega_2} m_j^2 e^{iz_j x + 8iz_j^3 t} \right) \\ & + \frac{1}{2\pi} \int_{\partial\Omega_2} c(k) e^{ikx + 8ik^3 t} dk + \frac{1}{2\pi} \int_{-\infty}^\infty r(k) e^{ikx + 8ik^3 t} dk. \end{aligned} \tag{1.7}$$

Properties of scattering data

Now we introduce three sets of conditions on a set \mathcal{R} of numbers $k_1, \dots, k_n \in \mathbb{C}_+$, $z_1, \dots, z_m \in \Omega_2$ and functions $r(k)$, $k \in \mathbb{R}$, $c(k)$, $k \in \Omega_2$.

Condition A. *Conditions on $r(k)$, $k \in \mathbb{R}$:*

- $r(k) \in C^\infty(\mathbb{R})$, $r(-k) = \bar{r}(k)$; $r(k) = O(k^{-1})$, as $k \rightarrow \infty$; $|r(k)| < 1$, if $\lambda = 1$.
- $r(k) = \frac{-s_2^-(k)}{s_2^+(k)}$, where $s_2^-(k)$ is analytic in $k \in \mathbb{C}_-$, $s_2^+(k)$ is analytic for $k \in \mathbb{C}_+$ and has the form:

$$s_2^+(k) = \left(\prod_{j=1}^n \frac{k - k_j}{k - \bar{k}_j} \right)^{\frac{1-\lambda}{2}} \exp \left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 - \lambda |r(\mu)|^2) d\mu}{\mu - k} \right], \quad k \in \mathbb{C}_+.$$

- The function given by

$$\int_{-\infty}^{\infty} r(k) e^{ik(x+8k^2t)} dk$$

is C^∞ in x, t for $x > 0$ and $t \geq 0$.

Condition B. *Conditions on $\mathcal{K} = \{k_1, \dots, k_n \in \mathbb{C}_+; z_1, \dots, z_m \in \Omega_2\}$:*

- If $\lambda = 1$, then $\mathcal{K} = \emptyset$.
- If $\lambda = -1$, then \mathcal{K} satisfies symmetry conditions:

$$\begin{aligned} k_j = i\kappa_j, \quad 1 \leq j \leq n_1 \leq n = n_1 + 2n_2 & \quad k_{n_1+l} = -\bar{k}_{n_1+n_2+l}, \quad 1 \leq l \leq n_2 \\ z_j = i\mu_j, \quad 1 \leq j \leq m_1 \leq m = m_1 + 2m_2, & \quad z_{m_1+l} = -\bar{z}_{m_1+m_2+l}, \quad 1 \leq l \leq m_2. \end{aligned}$$

Condition C. *Conditions on $c(k)$, $k \in \Omega_2$:*

- If $\lambda = 1$, then $c(k)$ is analytic in $k \in \mathbb{C}_+$ ($T < \infty$) or in Ω_2 ($T = \infty$) and it is bounded on $\overline{\Omega}_2$.
- If $\lambda = -1$, then $c(k)$ is meromorphic in the half-plane \mathbb{C}_+ ($T < \infty$) or in Ω_2 ($T = \infty$), where it has poles at z_1, z_2, \dots, z_m .
- $c(k) = -\bar{c}(-\bar{k})$, and $c(k) \rightarrow 0$, $k \rightarrow \infty$, $k \in \overline{\mathbb{C}}_+$ or $k \in \overline{\Omega}_2$.
- $c(k)$ has C^∞ boundary values on \mathbb{R} or $\partial\Omega_2$.
- If $T = \infty$, then $\frac{d^n c(k)}{dk^n} \Big|_{k=0} = -\frac{d^n r(k)}{dk^n} \Big|_{k=0}$, $n = 0, 1, 2, \dots$
- The function given by

$$\int_{\partial\Omega_2} c(k) e^{8ik^3t} dk + \int_{-\infty}^{\infty} r(k) e^{8ik^3t} dk$$

is C^∞ in t .

1.4 Main theorem

The main result is that properties A–B–C are characteristic:

Theorem. *Conditions A–B–C on*

$$\mathcal{R} = \{k_1, \dots, k_n \in \mathbb{C}_+; z_1, \dots, z_m \in \Omega_2; r(k), k \in \mathbb{R}; c(k), k \in \Omega_2\}$$

characterize the scattering data of the compatible eigenvalue problem (1.2)–(1.3) for x - and t -equations defined by a solution $q(x, t)$ of the mKdV equation (1.1) with initial function $u(x) \in \mathcal{S}(\mathbb{R}_+)$ and boundary values $v(t), v_1(t), v_2(t) \in C^\infty[0, T]$ if $T < \infty$, or $v(t), v_1(t), v_2(t) \in \mathcal{S}(\mathbb{R}_+)$ if $T = \infty$.

Remark. The third item in condition A (about smoothness of the integral with respect to x and t) is fulfilled if the initial and boundary functions obey the following relations :

$$\left. \frac{d^n}{dx^n} u(x) \right|_{x=0} = \left. \frac{d^n}{dt^n} v(t) \right|_{t=0} = \left. \frac{d^n}{dt^n} v_1(t) \right|_{t=0} = \left. \frac{d^n}{dt^n} v_2(t) \right|_{t=0} = 0 \quad (1.8)$$

for any $n \geq 0$. In this case the reflection coefficient $r(k)$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$. Under these assumptions the last item of condition C is also fulfilled.

Remark. If $u(x) \equiv 0$, conditions A are trivial: $r(k) \equiv 0$, $a(k) \equiv 1$, $\{k_1, \dots, k_n\} = \emptyset$. There remain only conditions B and C. They mean that any solution of the mKdV equation in the half-strip or quarter plane with zero initial function is parametrized by only one function $c(k) = -\bar{c}(-\bar{k})$, analytic ($\lambda = 1$) or meromorphic ($\lambda = -1$) in Ω_2 with poles at z_1, \dots, z_m .

2 Basic solutions of compatible x - and t -equations

Let us write the x - and t -equations in the form

$$W_x = U(x, t, k)W, \quad (2.1)$$

$$W_t = V(x, t, k)W, \quad (2.2)$$

where $U(x, t, k)$ and $V(x, t, k)$ are matrices given by

$$U(x, t, k) = Q(x, t) - ik\sigma_3,$$

$$V(x, t, k) = 2Q^3(x, t) - Q_{xx} - 2ik(Q^2(x, t) + Q_x(x, t))\sigma_3 + 4k^2Q(x, t) - 4ik^3\sigma_3.$$

Lemma 1. *Let the system (2.1), (2.2) be compatible for all k . Let $W(x, t, k)$ satisfy the x -equation (2.1) for all t , and let $W(x_0, t, k)$ satisfy the t -equation (2.2) for some $x = x_0$ (including the case $x_0 = \infty$). Then $W(x, t, k)$ satisfies the t -equation for all x .*

Proof. See e.g. [9]. □

Notations. The over-bar denotes the complex conjugation. \mathbb{C}_\pm denotes the upper (lower) complex half plane. If $A = \begin{pmatrix} A^- & A^+ \end{pmatrix}$ denotes a 2×2 matrix, the vectors A^\mp denote the first and second columns of A . We also denote $[A, B] = AB - BA$.

In this section we shall introduce basic solutions of compatible x - and t -equations.

2.1 First basic solution

The first basic solution is a matrix-valued Jost solution of the x -equation (1.2). It has the triangular integral representation (see e.g. [19])

$$\Psi(x, t, k) = \left(e^{-ikx\sigma_3} + \int_x^\infty K(x, y, t) e^{-iky\sigma_3} dy \right) e^{-4ik^3 t \sigma_3}, \quad (2.3)$$

where real-valued matrix $K(x, y, t)$ has the form

$$K(x, y, t) = \begin{pmatrix} K_1(x, y, t) & \lambda K_2(x, y, t) \\ K_2(x, y, t) & K_1(x, y, t) \end{pmatrix}$$

with entries in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$ and rapidly decreasing as $x + y \rightarrow \infty$ for any $t \in \mathbb{R}_+$. Matrices $K(x, x, t)$ and $Q(x, t)$ are connected by the relation:

$$[\sigma_3, K(x, x, t)] = Q(x, t)\sigma_3. \quad (2.4)$$

This last equality yields important formula (1.4) for the solution $q(x, t)$ of the modified Korteweg-de Vries equation. The matrix $\Psi(x, t, k)$ satisfies the x -equation (1.2) and it satisfies the t -equation (1.3) with $x = \infty$, because the matrix $e^{-ik(x+4k^2t)\sigma_3}$ is a solution of both equations (1.2) and (1.3) with $Q(x, t) \equiv 0$. Lemma 1 implies that $\Psi(x, t, k)$ satisfies the t -equation for any $x \in \mathbb{R}_+$ due to the compatibility of the x - and t -equations.

The triangular integral representation (2.3) and Lemma 1 imply the following properties of the matrix-valued Jost solution $\Psi(x, t, k)$ (cf. [19]):

Properties of the first basic solution

1. $\Psi(x, t, k)$ satisfies the x - and t -equations (1.2)-(1.3).
2. $\Psi(x, t, k) = \Lambda \bar{\Psi}(x, t, k) \Lambda^{-1}$ for $k \in \mathbb{R}$, $\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$.
3. $\det \Psi(x, t, k) \equiv 1$ for $k \in \mathbb{R}$.
4. $(x, t, k) \mapsto \Psi(x, t, k) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$
5. $\Psi^+(x, t, k)$ is analytic in $k \in \mathbb{C}_+$, $\Psi^-(x, t, k)$ is analytic in $k \in \mathbb{C}_-$.
6. $\Psi^\pm(x, t, k) = \overline{\Psi^\pm(x, t, -\bar{k})}$
7. For $k \rightarrow \infty$,

$$\begin{aligned} e^{ikx+4ik^3t} \Psi^-(x, t, k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k^{-1}) \quad \text{if } \operatorname{Im} k \leq 0, \\ e^{-ikx-4ik^3t} \Psi^+(x, t, k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(k^{-1}) \quad \text{if } \operatorname{Im} k \geq 0. \end{aligned}$$

2.2 Second basic solution

Now let us introduce the second basic solution $\Phi(x, t, k)$ of the x - and t -equations which satisfies the initial condition

$$\Phi(0, 0, k) = \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.5)$$

It can be represented as a product of two matrices:

$$\Phi(x, t, k) = \varphi(x, t, k) \hat{\varphi}(t, k), \quad (2.6)$$

where $\varphi(x, t, k)$ satisfies the x -equation under the condition $\varphi(0, t, k) = \sigma_0$, and $\hat{\varphi}(t, k)$ satisfies the t -equation with $x = 0$ under initial condition $\hat{\varphi}(0, k) = \sigma_0$. Lemma 1 implies that $\Phi(x, t, k)$ is a compatible solution of the x - and t -equations. The existence of the solution $\varphi(x, t, k)$ and its representation

$$\varphi(x, t, k) = e^{-ikx\sigma_3} + \int_{-x}^x A(x, y, t) e^{-iky\sigma_3} dy \quad (2.7)$$

by some integral kernel $A(x, y, t)$ are proved in [9]. The matrix $\hat{\varphi}(t, k)$ can be found as solution of the Volterra integral equation:

$$\hat{\varphi}(t, k) = e^{-4ik^3 t \sigma_3} + \int_0^t e^{4ik^3(\tau-t)} \hat{Q}(0, \tau, k) \hat{\varphi}(\tau, k) d\tau, \quad (2.8)$$

where

$$\hat{Q}(0, t, k) = \begin{pmatrix} -2i\lambda k v^2(t) & 2\lambda v^3(t) + 4k^2 v(t) + 2ikv_1(t) - v_2(t) \\ 2v^3(t) + 4\lambda k^2 v(t) - 2i\lambda k v_1(t) - \lambda v_2(t) & 2i\lambda k v^2(t) \end{pmatrix}$$

Besides $\hat{\varphi}(t, k)$ has the integral representation:

$$\begin{aligned} \hat{\varphi}(t, k) = & e^{-4ik^3 t \sigma_3} + \int_{-t}^t B(t, s) e^{-4ik^3 s \sigma_3} ds \\ & + ik \int_{-t}^t C(t, s) e^{-4ik^3 s \sigma_3} ds + k^2 \int_{-t}^t D(t, s) e^{-4ik^3 s \sigma_3} ds, \end{aligned} \quad (2.9)$$

which will be used below. The proof of this triangular representation can be done by the same way as in [9]. In the present case the matrix-valued real functions $A(x, y, t)$, $B(t, s)$, $C(t, s)$ and $D(t, s)$ are C^∞ and bounded in x, y, t, s .

The triangular integral representations (2.6)-(2.9) yield the following properties of the solution $\Phi(x, t, k)$:

Properties of the second basic solution

1. $\Phi(x, t, k)$ is a solution of the x - and t -equations.
2. $\Phi(x, t, k) = \Lambda \bar{\Phi}(x, t, \bar{k}) \Lambda^{-1}$ for any $k \in \mathbb{C}$.

3. $\Phi(x, t, k) = \bar{\Phi}(x, t, -\bar{k})$ for any $k \in \mathbb{C}$.
4. $\det \Phi(x, t, k) \equiv 1$ for any $k \in \mathbb{C}$.
5. $(x, t, k) \mapsto \Phi(x, t, k) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C})$.
6. $\Phi(x, t, k)$ is analytic (entire) in $k \in \mathbb{C}$.
7. For $k \in \mathbb{C}$, $k \rightarrow \infty$,

$$\Phi(x, t, k) = \left[I + O(k^{-1}) + O\left(\frac{e^{2ikx\sigma_3}}{k}\right) + O\left(\frac{e^{8ik^3t\sigma_3}}{k}\right) \right] e^{-ik(x+4k^2t)\sigma_3}.$$

8. For $k \in \Omega_1 \cup \Omega_3$, $k \rightarrow \infty$,

$$e^{ikx+4ik^3t}\Phi^-(x, t, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k^{-1}).$$

The last asymptotic relation can be easily proved using large k asymptotics for the functions $\varphi^\mp(x, t, k)$ and $\hat{\varphi}^-(t, k)$.

2.3 Third basic solution

Let $\Sigma = \{k \in \mathbb{C} \mid \operatorname{Im} k^3 = 0\}$ as above and let $\hat{\Psi}(t, k)$ be a solution of the Volterra integral equation

$$\hat{\Psi}(t, k) = e^{-4ik^3t\sigma_3} - \int_t^\infty e^{-4ik^3(\tau-t)\sigma_3} \hat{Q}(0, \tau, k) \hat{\Psi}(\tau, k) d\tau, \quad k \in \Sigma,$$

where $\hat{Q}(0, t, k)$ is as in (2.8) and $\hat{Q}(0, t, k) \equiv 0$ for $t > T$ if $T < \infty$, that means the matrix $\hat{\Psi}(t, k)$ satisfies the t -equation with $x = 0$ under the asymptotic condition $\hat{\Psi}(t, k) = e^{-4ik^3t\sigma_3} + o(1)$ as $t \rightarrow \infty$. Again, for $\hat{\Psi}(t, k)$ the triangular integral representation

$$\begin{aligned} \hat{\Psi}(t, k) &= e^{-4ik^3t\sigma_3} + \int_t^\infty L(t, s) e^{-4ik^3s\sigma_3} ds \\ &\quad + ik \int_t^\infty M(t, s) e^{-4ik^3s\sigma_3} ds + k^2 \int_t^\infty N(t, s) e^{-4ik^3s\sigma_3} ds \end{aligned} \quad (2.10)$$

can be obtained as in [9]. Here the matrix-valued real functions $L(t, s)$, $M(t, s)$ and $N(t, s)$ are C^∞ and bounded in t, s , and they vanish for $s > 2T - t$ if $T < \infty$. We introduce the matrix

$$Y(x, t, k) = \varphi(x, t, k) \hat{\Psi}(t, k), \quad k \in \Sigma, \quad (2.11)$$

where $\varphi(x, t, k)$ is as in (2.7). Lemma 1 implies that $Y(x, t, k)$ is a solution of the x - and t -equations with

$$\det Y(x, t, k) = 1 \text{ for } k \in \Sigma.$$

For $k \notin \Sigma$ the function $\hat{\Psi}(t, k)$, hence also $Y(x, t, k)$, is unbounded in $t \in \mathbb{R}_+$. Since the integral equation is of Volterra type with $\tau \in (t, \infty)$, the first column $Y^-(x, t, k)$ is analytic in $k \in \Omega_2 \cup \Omega_4 \cup \Omega_6$ and the second column $Y^+(x, t, k)$ is analytic in $k \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ or $Y(x, t, k)$ is an entire matrix-valued function if $T < \infty$.

The properties of the solution $Y(x, t, k)$ follow from the triangular integral representations (2.7) and (2.10):

Properties of the third basic solution

1. $Y(x, t, k)$ satisfies the x - and t -equations.
2. $Y(x, t, k) = \Lambda \bar{Y}(x, t, \bar{k}) \Lambda^{-1}$ for $k \in \Sigma$.
3. $\det Y(x, t, k) = 1$ for $k \in \Sigma$.
4. $(x, t, k) \mapsto Y(x, t, k) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \Sigma)$.
5. $Y^+(x, t, k)$ is analytic in $k \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, $Y^-(x, t, k)$ is analytic in $k \in \Omega_2 \cup \Omega_4 \cup \Omega_6$ or they are entire if $T < \infty$.
6. $\bar{Y}^-(x, t, -\bar{k}) = Y^-(x, t, k)$, $k \in \Omega_2 \cup \Omega_4 \cup \Omega_6$.
7. $\bar{Y}^+(x, t, -\bar{k}) = Y^+(x, t, k)$, $k \in \Omega_1 \cup \Omega_3 \cup \Omega_5$.
8. For $k \rightarrow \infty$, $k \in \Omega_2$,

$$e^{ikx+4ik^3t} Y^-(x, t, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k^{-1}).$$

3 Analysis of the direct scattering problem

The basic solutions we have introduced are clearly linearly dependent:

$$\begin{aligned} \Phi(x, t, k) &= \Psi(x, t, k) S(k), \\ Y(x, t, k) &= \Phi(x, t, k) P(k), \\ Y(x, t, k) &= \Psi(x, t, k) R(k). \end{aligned} \tag{3.1}$$

The matrices $S(k)$, $P(k)$ and $R(k)$ depend neither on x nor on t because by virtue of the x -equation they do not depend on x , and by virtue of the t -equation they do not depend on t . Hence:

$$\begin{aligned} S(k) &= \Psi^{-1}(0, 0, k), \quad k \in \mathbb{R}; \\ P(k) &= Y(0, 0, k), \quad k \in \Sigma; \\ R(k) &= S(k) P(k), \quad k \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Let us study the properties of these “scattering” (transition) matrices.

Properties of the scattering matrix $S(k)$

They follow from the scattering problem for the x -equation with $t = 0$. Indeed, consider the problem on the whole x -line by putting

$$q(x, 0) = \hat{u}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ u(x) & \text{for } x \in [0, \infty). \end{cases}$$

Let $\tilde{\Psi}(x, k)$ be the Jost solution [19] normalized by

$$\tilde{\Psi}(x, k) = e^{-ikx\sigma_3} \text{ for } x < 0$$

and let $\tilde{T}(k)$ be the transition matrix for that case, i.e.

$$\tilde{\Psi}(x, k) = \Psi(x, 0, k)\tilde{T}(k).$$

Putting $x = 0$ we find $S(k) \equiv \tilde{T}(k)$. Hence the “scattering” matrix $S(k)$ has all properties of the transition matrix $\tilde{T}(k)$ [19]:

- $S(k) = \Lambda \bar{S}(k) \Lambda^{-1}$ for $k \in \mathbb{R}$.
- $\det S(k) \equiv 1$ for $k \in \mathbb{R}$.
- $S(k) \in C^\infty(\mathbb{R})$.

For the half-line case there are additional properties:

- $S(k) = \begin{pmatrix} s_2^+(k) & -s_1^+(k) \\ -s_2^-(k) & s_1^-(k) \end{pmatrix}$ where $s_j^+(k) = \Psi_j^+(0, 0, k)$;
- $(s_2^+(k) \quad -s_1^+(k))$ is analytic in $k \in \mathbb{C}_+$ and $s_j^+(k) = \overline{s_j^+(-\bar{k})}$;
- $(-s_2^-(k) \quad s_1^-(k))$ is analytic¹ in $k \in \mathbb{C}_-$ and $s_j^-(k) = \overline{s_j^-(-\bar{k})}$;
- If $k \in \mathbb{C}_+$ and $k \rightarrow \infty$,

$$s_2^+(k) = 1 + O(k^{-1}), \quad s_1^+(k) = O(k^{-1}). \quad (3.3)$$

Let us prove some integral representations for $s_1^+(k)$ and $s_2^+(k)$. We use the limit formulas

$$s_1^+(k) = \lim_{x \rightarrow -\infty} e^{-ikx} \Psi_1^+(x, 0, k), \quad s_2^+(k) = \lim_{x \rightarrow -\infty} e^{-ikx} \Psi_2^+(x, 0, k),$$

which follow from the definition of the matrix $S(k)$. If one puts

$$\chi_j(x, k) = e^{-ikx} \Psi_j^+(x, 0, k)$$

¹For an arbitrary function $\hat{u}(x)$, $x \in \mathbb{R}$ these analytic properties do not hold, $s_2^+(k)$ and $s_1^-(k)$ are only analytic in $k \in \mathbb{C}_+$ and $k \in \mathbb{C}_-$ respectively. In our case $\hat{u}(x) \equiv 0$ for $x < 0$ and therefore $s_1^+(k)$ and $s_2^-(k)$ are also analytic in $k \in \mathbb{C}_+$ and $k \in \mathbb{C}_-$ respectively.

the x -equation yields

$$\begin{aligned}\chi_1' + 2ik\chi_1 &= \hat{u}(x)\chi_2 & \chi_1(x, k) &\rightarrow 0 \text{ as } x \rightarrow +\infty \\ \chi_2' &= -\hat{u}(x)\chi_1 & \chi_2(x, k) &\rightarrow 1 \text{ as } x \rightarrow +\infty.\end{aligned}$$

Then, by integration:

$$\begin{aligned}\chi_1(x, k) &= -\int_x^\infty e^{2ik(y-x)} \hat{u}(y) \chi_2(y, k) dy \\ \chi_2(x, k) &= 1 + \int_x^\infty \hat{u}(y) \chi_1(y, k) dy,\end{aligned}$$

therefore

$$s_1^+(k) = -\int_0^\infty u(x) e^{2ikx} dx - \int_0^\infty u(x) e^{2ikx} dx \int_0^\infty K_1(x, x+y) e^{iky} dy \quad (3.4)$$

$$s_2^+(k) = 1 - \int_0^\infty e^{ikx} dx \int_0^\infty u(y) K_2(y, y+x) dy, \quad (3.5)$$

where $K_1(x, y)$ and $K_2(x, y)$ are entries of the kernel of triangular integral transformation (2.3). The last two formulas allow to find the large- k asymptotic expansions at any order of $s_1^+(k)$ and $s_2^+(k)$ and to obtain (3.3), in particular, which is exact (precise) if $u(0) \neq 0$.

The matrix $S(k) = \Psi^{-1}(0, 0, k)$ is determined by $u(x) \in \mathcal{S}(\mathbb{R}_+)$. The entries of this matrix are not independent and can be recovered from one known function. Let

$$s(k) \equiv \frac{s_1^+(k)}{s_2^+(k)}$$

be given and let

$$\Sigma_d^{\text{ic}} = \{k_1, \dots, k_n \in \mathbb{C}_+ \mid s_2^+(k_j) = 0\},$$

be the set of zeros of the analytic function $s_2^+(k)$, which is finite because $s_2^+(k) \rightarrow 1$ as $k \rightarrow \infty$ and we have supposed that $s_2^+(k) \neq 0$ for any $k \in \mathbb{R}$. Since $\det S(k) \equiv 1$, then $|s_2^+(k)|^2 - \lambda |s_1^+(k)|^2 \equiv 1$ for any $k \in \mathbb{R}$. This identity yields the well-known formula:

$$s_2^+(k) = \left(\prod_{k_j \in \mathbb{C}_+} \frac{k - k_j}{k - \bar{k}_j} \right)^{\frac{1-\lambda}{2}} \exp \left\{ \frac{i}{2\pi} \int_{-\infty}^\infty \frac{\log[1 - \mu |s(\mu)|^2] d\mu}{\mu - k} \right\}, \quad (3.6)$$

The remaining entries of $S(k)$ are also recovered:

$$s_1^+(k) = s(k) s_2^+(k), \quad s_2^-(k) = \lambda \bar{s}_1^+(\bar{k}), \quad s_1^-(k) = \bar{s}_2^+(\bar{k}).$$

So, if $\lambda = 1$, the function $s_2^+(k)$ has no zeros at all and the set Σ_d^{ic} is empty. It follows from the self-adjointness of the x -equation (1.2) and the obvious inequality: $|s_2^+(k)| \geq 1$ for $k \in \mathbb{R}$. If $\lambda = -1$ then $s_2^+(k)$ may vanish at some points $k_j \in \mathbb{C}_+$. Since $u(x)$ is real-valued, Σ_d^{ic} is symmetric with respect to the imaginary axis. We can enumerate the k_j 's in such a way that

- $k_j = i\kappa_j$, $\kappa_j > 0$ for $j = 1, \dots, n_1 \leq n$ with $n = n_1 + 2n_2$,
- $k_{n_1+l} = -\bar{k}_{n_1+n_2+l}$ for $l = 1, \dots, n_2$.

Moreover, these zeros can be multiple and there can exist limit points on the real line \mathbb{R} [19].

To avoid this difficulties we shall consider a subset $\mathcal{S}_0(\mathbb{R}_+)$ of functions $u(x) \in \mathcal{S}(\mathbb{R}_+)$ for which $s_2^+(k)$ has a finite number of zeros k_1, \dots, k_n in \mathbb{C}_+ , all of multiplicity 1, i.e. $\dot{s}_2^+(k_j) \neq 0$, and $s_2^+(k) \neq 0$ for every $k \in \mathbb{R}$.

Let us briefly discuss the discrete spectrum of the x -problem, which may appear when $\lambda = -1$. The main relation of the x -scattering problem is

$$\frac{1}{s_2^+(k)} \Phi^-(x, t, k) = \Psi^-(x, t, k) + r(k) \Psi^+(x, t, k) \text{ for } k \in \mathbb{R}, \quad (3.7)$$

where

$$r(k) = -\frac{s_2^-(k)}{s_2^+(k)}. \quad (3.8)$$

$F(x, t, k) = \Phi^-(x, t, k)/s_2^+(k)$ is analytic in $k \in \mathbb{C}_+$ except for $\Sigma_d^{\text{ic}} = \{k_1, \dots, k_n\}$, where it has poles. We have

$$s_2^+(k_j) = \det \begin{bmatrix} \Phi^-(x, t, k_j) & \Psi^+(x, t, k_j) \end{bmatrix} = 0,$$

then $\Phi^-(x, t, k_j) = \gamma_j^1 \Psi^+(x, t, k_j)$. Hence,

$$\text{Res}_{k=k_j} F(x, t, k) = c_j^1 \Psi^+(x, t, k_j)$$

with

$$c_j^1 = \frac{\gamma_j^1}{\dot{s}_2^+(k_j)} \quad \text{and} \quad \gamma_j^1 = \frac{1}{s_1^+(k_j)}, \quad j = 1, \dots, n.$$

The dot denotes differentiation with respect to k . Note that $s_1^+(k_j) \neq 0$ because otherwise we come to a contradiction: $\Psi_+(x, t, k_j) \equiv 0$ since $\Psi_1^+(0, 0, k_j) = s_1^+(k_j) = 0$ and $\Psi_2^+(0, 0, k_j) = s_2^+(k_j) = 0$. We also assume all zeros are simple, i.e. $\dot{s}_2^+(k_j) \neq 0$.

Using asymptotics of the function $\Phi^-(x, t, k)$ at $k = \infty$, for $k \in \Omega_1 \cup \Omega_3$, we find

$$F(x, t, k) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(|k|^{-1}) \right] e^{-ikx - 4ik^3 t} \text{ for } |k| \rightarrow \infty, \quad k \in \Omega_1 \cup \Omega_3, \quad (3.9)$$

that will be used below. So, we come to the following (cf. conditions A and B):

Properties of $r(k)$, $t(k)$ and k_j

- The reflection coefficient $r(k)$ belongs to $C^\infty(\mathbb{R})$, $r(-k) = \bar{r}(k)$ and $r(k) = O(k^{-1})$ as $k \rightarrow \infty$. It is the ratio of two functions $-s_2^-(k)$ and $s_2^+(k)$ analytic in $k \in \mathbb{C}_-$ and $k \in \mathbb{C}_+$ respectively, and $|r(k)| < 1$ if $\lambda = 1$.

- The transition coefficient $t(k) = [s_2^+(k)]^{-1}$ is represented through formula (3.6), where $|s(\mu)| = |r(\mu)|$.
- If $\lambda = -1$, $k_i \neq k_j$ for $i \neq j$ and $\text{Im } k_j > 0$, $j = 1, \dots, n$, with:
 $k_j = i\kappa_j$, $1 \leq j \leq n_1 \leq n = n_1 + 2n_2$, and $k_{n_1+l} = -\bar{k}_{n_1+n_2+l}$, $1 \leq l \leq n_2$.

These properties follow from the x -scattering problem on the whole line. We take into account that for the half-line case the function $s_2^-(k) = -\bar{s}_1^+(\bar{k})$ is analytic in $k \in \mathbb{C}_-$ and the constants γ_j^1 are not independent parameters (that takes place for the whole line), and they are evaluated by means of the function $s_1^+(k)$ at k_j : $\gamma_j^1 = 1/s_1^+(k_j)$.

Properties of the scattering matrix $P(k)$

They follow from the defining relation, i.e. from (3.2), (2.11), (2.10):

$$P(k) = I + \int_0^\infty L(t, s) e^{-4ik^3 s \sigma_3} ds + k \int_0^\infty M(t, s) e^{-4ik^3 s \sigma_3} ds + k^2 \int_0^\infty N(t, s) e^{-4ik^3 s \sigma_3} ds.$$

It is easy to find the following properties:

- $P(k) = \Lambda \bar{P}(\bar{k}) \Lambda^{-1}$ for $k \in \Sigma$.
- $\det P(k) \equiv 1$ for $k \in \Sigma$.
- $P(k)$ is C^∞ in $k \in \Sigma$.
- If $T < \infty$ the matrix-valued function $P(k)$ is entire in $k \in \mathbb{C}$.
- If $T = \infty$ the vector-function $P^+(k)$ is analytic in $k \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, and $P^-(k)$ is analytic in $k \in \Omega_2 \cup \Omega_4 \cup \Omega_6$.
- $P^\pm(k) = \overline{P^\pm(-\bar{k})}$.
- $P(k) = \sigma_0 + O(k^{-1})$, $k \in \Sigma$, $k \rightarrow \infty$.

Properties of the scattering matrix $R(k)$

Below we need to study properties of the “scattering” matrix $R(k)$ introduced by equations (3.1)-(3.2). The matrix $R(k)$ has the following form

$$R(k) = \begin{pmatrix} r_1^-(k) & r_1^+(k) \\ r_2^-(k) & r_2^+(k) \end{pmatrix}, \quad r_1^+(k) = -\bar{r}_2^-(\bar{k}), \quad r_2^+(k) = \bar{r}_1^-(\bar{k}) \quad \text{for } k \in \Sigma$$

with

$$r_1^-(k) = p_1^-(k) s_2^+(k) - p_2^-(k) s_1^+(k) \quad (3.10)$$

analytic in $k \in \mathbb{C}_+$ (if $T < \infty$) and $k \in \Omega_2$ (if $T = \infty$). Hence $r_2^+(k)$ is analytic in $k \in \mathbb{C}_-$ (if $T < \infty$) and $k \in \Omega_5$ (if $T = \infty$). Furthermore

$$r_2^-(k) = p_2^-(k) s_1^-(k) - p_1^-(k) s_2^-(k) \quad (3.11)$$

is analytic in $k \in \mathbb{C}_-$ (for $T < \infty$) and $k \in \Omega_4 \cup \Omega_6$ (for $T = \infty$), hence $r_1^+(k)$ is analytic in $k \in \mathbb{C}_+$ (for $T < \infty$) and $k \in \Omega_1 \cup \Omega_3$ (for $T = \infty$). In the domains of analyticity we have the following symmetry properties :

$$r_1^\pm(k) = \overline{r_1^\pm(-\bar{k})} \quad \text{and} \quad r_2^\pm(k) = \overline{r_2^\pm(-\bar{k})}.$$

From (3.1) we derive

$$Y^+(x, t, k) = r_1^+(k)\Psi^-(x, t, k) + r_2^+(k)\Psi^+(x, t, k)$$

with

$$r_1^+(k) = \det \begin{bmatrix} Y^+(x, t, k) & \Psi^+(x, t, k) \end{bmatrix}, \quad r_2^+(k) = \det \begin{bmatrix} \Psi^-(x, t, k) & Y^+(x, t, k) \end{bmatrix}.$$

Let us put $x = 0$, and $k = k_1 + ik_2 \in \Omega_1 \cup \Omega_3$. Using (2.3), (2.11), (2.10) for large t we obtain

$$|r_1^+(k)| \leq C_1(k) \exp[8t(K - (3k_1^2 - k_2^2)k_2)]$$

where $C_1(k)$ is independent of t , and

$$K = \max_{1 \leq j \leq n} [(3 \operatorname{Re}^2 k_j - \operatorname{Im}^2 k_j) \operatorname{Im} k_j],$$

where $k_j \in \Omega_1 \cup \Omega_3$ is an eigenvalue of the x -scattering problem. Taking into account the analyticity of the function $r_1^+(k)$ for $k \in \Omega_1 \cup \Omega_3$, choosing a large enough k and putting $t \rightarrow \infty$ (if $T = \infty$) we find $r_1^+(k) \equiv 0$ for any $k \in \Omega_1 \cup \Omega_3$, hence $r_2^-(k) \equiv 0$ for any $k \in \Omega_4 \cup \Omega_6$. So, we come to the main property of the compatible scattering problem for x - and t -equations.

- If $T = \infty$ the “scattering” matrix $R(k)$ is diagonal:

$$R(k) = \begin{pmatrix} \rho_-(k) & 0 \\ 0 & \rho_+(k) \end{pmatrix} \quad \text{for } k \in \mathbb{R}$$

with

$$\rho_+(k) = \frac{p_2^+(k)}{s_2^+(k)} = \frac{p_1^+(k)}{s_1^+(k)}, \quad \rho_-(k) = \frac{p_1^-(k)}{s_1^-(k)} = \frac{p_2^-(k)}{s_2^-(k)}. \quad (3.12)$$

- We also have the important relation

$$\frac{p_1^+(k)}{p_2^+(k)} \equiv \frac{s_1^+(k)}{s_2^+(k)} \quad (3.13)$$

that says: the function $p(k) := p_1^+(k)/p_2^+(k)$ being meromorphic in the domain $\Omega_1 \cup \Omega_3$ has an analytic continuation into \mathbb{C}_+ up to the meromorphic function $s(k) = s_1^+(k)/s_2^+(k)$.

- If $T < \infty$, instead of (3.13) we have the so-called *global relation* [21] :

$$s_2^+(k)p_1^+(k, T) - s_1^+(k)p_2^+(k, T) = r_1^+(k, T),$$

where we write down the dependence on T to emphasize that functions $p_j^+(k)$ ($j = 1, 2$) and $r_1^+(k)$ really depend on T .

The asymptotic behavior of $S(k)$ and $P(k)$ yields the following asymptotic expansions :

$$r_1^-(k) = 1 + \frac{\rho_1}{k} + \dots, \quad r_2^-(k) = \frac{\omega_1}{k} + \frac{\omega_2}{k^2} + \dots \quad \text{for } k \rightarrow \pm\infty.$$

If $T = \infty$ then $r_2^-(k) \equiv r_1^+(k) \equiv 0$. Since $\det R(k) \equiv 1$, then $\rho_-(k)\rho_+(k) = |\rho_+(k)|^2 \equiv 1$. Hence, $\rho_{\pm}(k)$ can be written in the form:

$$\rho_{\pm}(k) = e^{\pm i\nu(k)} \quad k \in \mathbb{R} \quad (3.14)$$

with a real function $\nu(k)$ for $k \in \mathbb{R}$. The function $\nu(k)$ has an analytic continuation to the domain $\Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6$ which satisfies:

- $\nu(k) = \bar{\nu}(\bar{k})$, $\nu(k) = -\overline{\nu(-\bar{k})}$, and $\nu(k) \rightarrow 0$ as $k \rightarrow \infty$,

in view of the asymptotics of the functions $s_2^+(k)$ and $p_2^+(k)$. Indeed, in view of (3.12),

$$p_j^+(k) = \rho_+(k) s_j^+(k) \text{ for } j = 1, 2, \quad (3.15)$$

$\rho_+(k)$ must have poles at the points where $s_1^+(k)$ and $s_2^+(k)$ vanish. On the other hand, $s_1^+(k)$ and $s_2^+(k)$ must simultaneously vanish at poles in view of the analyticity of the functions $p_j^+(k)$ for $k \in \Omega_1 \cup \Omega_3$. Hence $\Psi^+(x, t, k)$ vanish identically if k is a pole, which is impossible. So $\rho_+(k)$ is analytic (without singularities) in $k \in \Omega_1 \cup \Omega_3$. Hence the functions $p_j^+(k)$ and $s_j^+(k)$ have a common set of zeros, possibly empty, in $\Omega_1 \cup \Omega_3$. The other statements about $\nu(k)$ are obvious.

So, for $r_1^-(k)$ which is analytic in $k \in \Omega_2$ we obtain:

$$r_1^-(k) = \frac{s_2^+(k)}{p_2^+(k)} = e^{-i\nu(k)} \text{ for } k \in \partial\Omega_2, \quad (3.16)$$

The last formula follows from relations:

$$r_1^-(k) = p_1^-(k)s_2^+(k) - p_2^-(k)s_1^+(k) \quad p_1^+(k)s_2^+(k) - p_2^+(k)s_1^+(k) = 0.$$

Hence the function $r_1^-(k)$ has an analytic continuation to the domain $\Omega_1 \cup \Omega_3$, where it coincides with the function $1/\rho_+(k)$. Therefore the function $r_1^-(k)$ does not vanish for $k \in \partial\Omega_2$, and its zeros are some points $z_j \in \Omega_2$. Let Σ_d^{bc} be the set of zeros of the function $r_1^-(k)$. As above we assume that the number of zeros is finite:

$$\Sigma_d^{\text{bc}} = \{z_1, \dots, z_m \in \Omega_2 \mid r_1^-(z_j) = 0\}.$$

We also assume all zeros are simple, i.e. $\dot{r}_1^-(z_j) \neq 0$. We have $\Sigma_d^{bc} = \emptyset$ if $\lambda = 1$. Let

$$\rho(k) := \frac{r_2^-(k)}{r_1^-(k)}.$$

The functions $\rho(k)$ and $r_1^-(k)$ are dependent. They satisfy the determinant relation

$$1 - \lambda |\rho(k)|^2 = \frac{1}{|r_1^-(k)|^2}, \quad k \in \mathbb{R}, \quad (3.17)$$

and $\rho(k) \equiv 0$ if $T = \infty$. They have the following properties:

- $\rho(k), r_1^-(k) \in C^\infty(\mathbb{R})$, and $\rho(-k) = \bar{\rho}(k)$, $r_1^-(k) = \bar{r}_1^-(-k)$,
- If $T = \infty$, $\rho(k) \equiv 0$ for $k \in \mathbb{R}$, and $r_1^-(k) = e^{-i\nu(k)}$ where $\nu(k)$ is described above.
- If $T < \infty$, then

$$r_1^-(k) = \left(\prod_{z_j \in \mathbb{C}_+} \frac{k - z_j}{k - \bar{z}_j} \right)^{\frac{1-\lambda}{2}} \exp \left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 - \lambda |\rho(s)|^2) ds}{s - k} \right], \quad k \in \mathbb{C}_+.$$

- The function

$$\rho(k) - r(k) = \frac{p_2^-(k)}{r_1^-(k) s_2^+(k)}$$

has an analytic continuation to \mathbb{C}_+ for $T < \infty$.

For $T = \infty$ the r.h.s. is analytic only in Ω_2 .

The last item follows from equations (3.10) and (3.11) which yield

$$\begin{aligned} p_1^-(k) &= r_2^-(k) s_1^+(k) + r_1^-(k) s_1^-(k) = r_1^-(k) [1/s_2^+(k) + s_1^+(k)(\rho(k) - r(k))], \\ p_2^-(k) &= r_2^-(k) s_2^+(k) + r_1^-(k) s_2^-(k) = r_1^-(k) s_2^+(k) [\rho(k) - r(k)] \text{ for } k \in \mathbb{R}. \end{aligned}$$

For $T < \infty$ the difference $\rho(k) - r(k)$ has an analytic continuation to \mathbb{C}_+ because the l.h.s. are analytic in $k \in \mathbb{C}_+$. Hence, the r.h.s. must have analytic continuations to \mathbb{C}_+ .

The second main relation of the compatible scattering problem is:

$$\begin{aligned} G(x, t, k) &= \frac{1}{r_1^-(k)} Y^-(x, t, k) \\ &= \begin{cases} \Psi^-(x, t, k) + \rho(k) \Psi^+(x, t, k) & \text{for } k \in \mathbb{R} \text{ if } T < \infty \\ \Psi^-(x, t, k) & \text{for } k \in \mathbb{R} \text{ if } T = \infty. \end{cases} \end{aligned} \quad (3.18)$$

The function $G(x, t, k)$ is analytic in $k \in \Omega_2$, for $k \neq z_j$ and the z_j 's are poles of G . If $r_1^-(z_j) = 0$ then $Y^-(x, t, z_j)$ and $\Psi^+(x, t, z_j)$ are linearly dependent:

$$Y^-(x, t, z_j) = \gamma_j^2 \Psi^+(x, t, z_j), \quad j = 1, \dots, m,$$

hence

$$\text{Res}_{k=z_j} G(x, t, k) = c_j^2 \Psi^+(x, t, z_j), \quad c_j^2 = \frac{\gamma_j^2}{\dot{r}_1^-(z_j)}$$

(the dot denotes differentiation with respect to k) with

$$\gamma_j^2 = \frac{p_1^-(z_j)}{s_1^+(z_j)} = \frac{p_2^-(z_j)}{s_2^+(z_j)}.$$

Using asymptotics of the function $Y^-(x, t, k)$ in the neighborhood of $k = \infty$ for $k \in \Omega_2$, we find

$$G(x, t, k) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(|k|^{-1}) \right] e^{-ikx - 4ik^3 t} \quad (3.19)$$

for $|k| \rightarrow \infty$, $k \in \Omega_2$.

This asymptotic formula will be used in the next section.

4 The main integral equations

The main relations of the compatible scattering problem follow from (3.1), (3.2) and (3.7), (3.18):

$$F(x, t, k) = \Psi^-(x, t, k) + r(k) \Psi^+(x, t, k) \text{ for } k \in \mathbb{R}, \quad (4.1)$$

$$G(x, t, k) = \Psi^-(x, t, k) + \rho(k) \Psi^+(x, t, k) \text{ for } k \in \mathbb{R}. \quad (4.2)$$

These relations give:

$$G(x, t, k) - F(x, t, k) = c(k) \Psi^+(x, t, k), \quad (4.3)$$

where $c(k)$ can be written as follows

$$c(k) = \rho(k) - r(k) = \frac{p_2^-(k)}{s_2^+(k) r_1^-(k)} \quad \text{for } \begin{cases} k \in \mathbb{C}_+ & \text{if } T < \infty, \\ k \in \Omega_2 & \text{if } T = \infty. \end{cases} \quad (4.4)$$

Properties of $c(k)$

Indeed, $c(k)$ is meromorphic in \mathbb{C}_+ if $T < \infty$, and in Ω_2 if $T = \infty$, and $c(k) = -\overline{c(-\bar{k})}$, because $p_2^-(k)$, $s_2^+(k)$ and $r_1^-(k)$ are analytic in $k \in \mathbb{C}_+$ if $T < \infty$, and in $k \in \Omega_2$ if $T = \infty$. Hence relation (4.3) is true for all $k \in \overline{\Omega_2}$. The function $c(k)$ has poles at the points z_j , where $s_2^+(z_j) = r_1^-(z_j) = 0$. Since the zeros of $s_2^+(k)$ and $r_1^-(k)$ are simple and in finite number, all poles of $c(k)$ are simple and also in finite number. Indeed, we only have to check the case $s_2^+(z_0) = r_1^-(z_0) = 0$. Due to (3.10) we also find $p_2^-(z_0) = 0$.

We have the following relation on $\partial\Omega_2$:

$$\frac{Y^-(x, t, k - 0)}{r_1^-(k - 0)} - \frac{\Phi^-(x, t, k + 0)}{s_2^+(k + 0)} = c(k) \Psi^+(x, t, k) \text{ for } k \in \partial\Omega_2. \quad (4.5)$$

To deduce the integral equations of the inverse scattering problem let us put

$$\begin{aligned} h^-(x, t, k) &= G(x, t, k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx-4ik^3t} \text{ for } k \in \partial\Omega_2 \\ h^+(x, t, k) &= F(x, t, k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx-4ik^3t} \text{ for } k \in \mathbb{R}. \end{aligned}$$

Let us consider the integral

$$J(x, y, t) = \frac{1}{2\pi} \int_{\partial\Omega_2} h^-(x, t, k) e^{iky+4ik^3t} dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} h^+(x, t, k) e^{iky+4ik^3t} dk.$$

Using equations (4.1), (4.2), (4.4), (2.3) we find

$$\begin{aligned} J(x, y, t) - \frac{1}{2\pi} \int_{\partial\Omega_2} h^-(x, t, k) e^{iky+4ik^3t} dk &= \\ &= \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (x, y, t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F_s(x+y, t) + \int_x^\infty \begin{pmatrix} \lambda K_2 \\ K_1 \end{pmatrix} (x, z, t) F_s(z+y, t) dz, \end{aligned}$$

where

$$F_s(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ik(x+y)+8ik^3t} dk.$$

On the other hand, using estimates (3.9) and (3.19) of $F(x, t, z)$ and $G(x, t, z)$ for large k , taking into account (4.3) (4.4), (4.5) and applying the Jordan lemma, we find

$$\begin{aligned} J(x, y, t) &= \frac{1-\lambda}{2} \left(i \sum_{\substack{k_j \in \Omega_1 \cup \Omega_3 \\ s_2^+(k_j)=0}} \text{Res}_{k=k_j} [h^+(x, t, k) e^{iky+4ik^3t}] \right. \\ &\quad \left. + i \sum_{\substack{z_j \in \Omega_2 \\ r_1^-(z_j)=0}} \text{Res}_{k=z_j} [h^-(x, t, k) e^{iky+4ik^3t}] \right) \\ &\quad - \frac{1}{2\pi} \int_{\partial\Omega_2} [G(x, t, k) - F(x, t, k)] e^{iky+4ik^3t} dk \\ &= \frac{1-\lambda}{2} \left(- \sum_{k_j \in \Omega_1} m_j^1 e^{ik_j y + 4ik_j^3 t} \Psi^+(x, t, k_j) - \sum_{k_j \in \Omega_3} m_j^3 e^{ik_j y + 4ik_j^3 t} \Psi^+(x, t, k_j) \right) \\ &\quad - \frac{1-\lambda}{2} \sum_{z_j \in \Omega_2} m_j^2 e^{iz_j y + 4iz_j^3 t} \Psi^+(x, t, z_j) - \frac{1}{2\pi} \int_{\partial\Omega_2} c(k) e^{iky+4ik^3t} \Psi^+(x, t, k) dk. \end{aligned}$$

Finally we have the following integral equations of the inverse scattering:

$$K_1(x, y, t) + \lambda \int_x^\infty K_2(x, z, t) H(z+y, t) dz = 0 \text{ for } 0 \leq x < y < \infty, \quad (4.6)$$

$$K_2(x, y, t) + H(x+y, t) + \int_x^\infty K_1(x, z, t) H(z+y, t) dz = 0 \quad (4.7)$$

with the kernel

$$H(x, t) = \frac{1 - \lambda}{2} \left(\sum_{k_j \in \Omega_1} m_j^1 e^{ik_j x + 8ik_j^3 t} + \sum_{z_j \in \Omega_2} m_j^2 e^{iz_j x + 8iz_j^3 t} + \sum_{k_j \in \Omega_3} m_j^3 e^{ik_j x + 8ik_j^3 t} \right) + \frac{1}{2\pi} \int_{\partial\Omega_2} c(k) e^{ikx + 8ik^3 t} dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx + 8ik^3 t} dk. \quad (4.8)$$

The coefficients m_j^1 and m_j^2 are given by

$$m_j^1 = [is_1^+(k_j) \dot{s}_2^+(k_j)]^{-1}, \\ m_j^2 = p_1^-(z_j) [is_1^+(z_j) \dot{r}_1^-(z_j)]^{-1} = p_2^-(z_j) [is_2^+(z_j) \dot{r}_1^-(z_j)]^{-1} = -i \operatorname{Res}_{k=z_j} c(k). \quad (4.9)$$

Using (4.7) for $y = x$ one can prove that $H(x, t) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and is rapidly decreasing in x , i.e. $H(x, t) = O(x^{-\infty})$, as $x \rightarrow \infty$, since $K_1(x, y, t)$ and $K_2(x, y, t)$ are in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$ and also rapidly decreasing as $x + y \rightarrow \infty$, and (4.7) is a Volterra integral equation with respect to the kernel $H(x, t)$. All terms in (4.8) are clearly C^∞ and of the Schwartz type except for the last term :

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx + 8ik^3 t} dk$$

because the reflection coefficient $r(k)$ vanish at infinity as $O(k^{-1})$ that follows from (3.3), (3.4). Thus we arrive to the requirement that this integral must be C^∞ in x, t (third item in condition A). It is easy to see that under additional condition (1.8) the reflection coefficient $r(k) \in \mathcal{S}(\mathbb{R})$, therefore the third item can be omitted because the corresponding integral is C^∞ . By assumption the boundary functions $q(0, t)$, $q'_x(0, t)$, $q''_{xx}(0, t)$ are C^∞ , then we obtain the following condition: the kernel $H(x, t)$ must be C^∞ in t when $x = 0$. Therefore the function

$$\int_{\partial\Omega_2} c(k) e^{8ik^3 t} dk + \int_{-\infty}^{\infty} r(k) e^{8ik^3 t} dk.$$

should be C^∞ with fast decay as $t \rightarrow \infty$. Thus we arrive to the properties given in the fifth item in condition C.

For any fixed $t \in \mathbb{R}_+$ the function $H(x, t)$ will be rapidly decreasing for $x \rightarrow \infty$. Indeed, using the method of steepest descent and integration by parts we see that $H(x, t) = O(x^{-\infty})$ because $c(k)$ and $r(k)$ are C^∞ and, according to their asymptotic behavior, they vanish at infinity as well as their derivatives of any order.

Remark. For $t = 0$ the kernel $H(x, t)|_{t=0}$ coincides with the kernel

$$H_0(x) = \sum_{k_j \in \mathbb{C}_+} m_j e^{ik_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk,$$

because in this case ($t = 0$) the integral over $\partial\Omega_2$ can be evaluated by using the residues of the function $c(k)$. After integration we find that $H(x, 0) = H_0(x)$. Then Marchenko integral equations with kernel $H_0(x)$ yield $q(x, 0) = u(x)$.

Now it is natural to introduce the set

$$\mathcal{R} = \{k_1, k_2, \dots, k_n \in \mathbb{C}_+; z_1, z_2, \dots, z_m \in \Omega_2; r(k), k \in \mathbb{R}; c(k), k \in \Omega_2\} \quad (4.10)$$

and to call it (see §1.4) the set of scattering data of the compatible eigenvalue problem for the pair of differential equations (1.2), (1.3) defined by $q(x, t)$ satisfying (1.1). The kernel $H(x, t)$ of the Marchenko equations is completely defined by the scattering data \mathcal{R} because the three missing coefficients m_j^1, m_j^2, m_j^3 (4.9) can be evaluated from the scattering data. Conditions A–B–C from the Introduction follow immediately from the properties proved above for the scattering data \mathcal{R} .

The properties given by conditions A–B–C are characteristic, i.e. they are sufficient to ensure that the system of numbers $k_1, \dots, k_n, z_1, \dots, z_m$ and functions $r(k), c(k)$, are the scattering data of compatible x - and t -equations (1.2), (1.3) with $q(x, t)$ satisfying the mKdV equation (1.1) with an initial function $u(x) \in \mathcal{S}(\mathbb{R}_+)$ and boundary values $v(t), v_1(t), v_2(t) \in C^\infty[0, T]$ if $T < \infty$, or $v(t), v_1(t), v_2(t) \in \mathcal{S}(\mathbb{R}_+)$ if $T = \infty$.

Anyway formula (1.4) and Marchenko integral equations (4.6), (4.7) represent a solution of the mKdV equation if the kernel (4.8) is sufficiently smooth and rapidly decreasing as $x \rightarrow \infty$. It follows from statements in Section 6.

5 Formulation of the Riemann-Hilbert problem

Here we give a formulation of the inverse scattering problem as a Riemann-Hilbert problem which will be used for proving that the solution $q(x, t)$ arising from Marchenko equations satisfies the boundary conditions. We recall that the equality $q(x, 0) = u(x)$ is already proved.

The main scattering relations (3.1) yield the following Riemann-Hilbert problem. Indeed, from (3.1) we derive :

$$\begin{aligned} \frac{\Phi^-(x, t, k)}{s_2^+(k)} &= \Psi^-(x, t, k) + r(k)\Psi^+(x, t, k), \quad k \in \mathbb{R}, \\ \frac{Y^-(x, t, k)}{r_1^-(k)} &= \Psi^-(x, t, k) + \rho(k)\Psi^+(x, t, k), \quad k \in \mathbb{R}, \end{aligned} \quad (5.1)$$

$$\begin{aligned} \frac{\Phi^+(x, t, k)}{s_1^-(k)} &= \Psi^+(x, t, k) + \lambda \bar{r}(k)\Psi^-(x, t, k), \quad k \in \mathbb{R}, \\ \frac{Y^+(x, t, k)}{r_2^+(k)} &= \Psi^+(x, t, k) + \lambda \bar{\rho}(k)\Psi^-(x, t, k), \quad k \in \mathbb{R}, \end{aligned} \quad (5.2)$$

$$\frac{Y^-(x, t, k)}{r_1^-(k)} - \frac{\Phi^-(x, t, k)}{s_2^+(k)} = c(k)\Psi^+(x, t, k), \quad k \in \partial\Omega_2, \quad (5.3)$$

$$\frac{Y^+(x, t, k)}{r_2^+(k)} - \frac{\Phi^+(x, t, k)}{s_1^-(k)} = \lambda \bar{c}(\bar{k})\Psi^-(x, t, k), \quad k \in \partial\Omega_5. \quad (5.4)$$

Let us define the sectionally meromorphic (analytic for $\lambda = 1$) matrix $M(k, x, t)$:

$$M(k, x, t) = \begin{cases} \begin{pmatrix} \frac{\Phi_1^-(x, t, k)e^{i\theta}}{s_2^+(k)} & \Psi_1^+(x, t, k)e^{-i\theta} \\ \frac{\Phi_2^-(x, t, k)e^{i\theta}}{s_2^+(k)} & \Psi_2^+(x, t, k)e^{-i\theta} \end{pmatrix} & k \in \Omega_1 \cup \Omega_3 \\ \begin{pmatrix} \frac{Y_1^-(x, t, k)e^{i\theta}}{r_1^-(k)} & \Psi_1^+(x, t, k)e^{-i\theta} \\ \frac{Y_2^-(x, t, k)e^{i\theta}}{r_1^-(k)} & \Psi_2^+(x, t, k)e^{-i\theta} \end{pmatrix} & k \in \Omega_2 \\ \begin{pmatrix} \Psi_1^-(x, t, k)e^{i\theta} & \frac{Y_1^+(x, t, k)e^{-i\theta}}{r_2^+(k)} \\ \Psi_2^-(x, t, k)e^{i\theta} & \frac{Y_2^+(x, t, k)e^{-i\theta}}{r_2^+(k)} \end{pmatrix} & k \in \Omega_5 \\ \begin{pmatrix} \Psi_1^-(x, t, k)e^{i\theta} & \frac{\Phi_1^+(x, t, k)e^{-i\theta}}{s_1^-(k)} \\ \Psi_2^-(x, t, k)e^{i\theta} & \frac{\Phi_2^+(x, t, k)e^{-i\theta}}{s_1^-(k)} \end{pmatrix} & k \in \Omega_4 \cup \Omega_6. \end{cases}$$

Here $\theta = \theta(x, t, k) = k(x + 4k^2t)$. Then we have the following Riemann-Hilbert problem

$$M_-(k, x, t) = M_+(k, x, t)J(k, x, t), \quad k \in \Sigma \quad (5.5)$$

on the contour $\Sigma = \{k \mid \text{Im } k^3 = 0\}$. The orientation on the contour Σ is chosen in such a way that the sign “+” (resp. “−”) corresponds to the left (resp. right) boundary values of the matrix $M(k, x, t)$ in the domains $\Omega_1, \Omega_3, \Omega_5$ marked by “+”, and in the domains $\Omega_2, \Omega_4, \Omega_6$ marked by “−”. The corresponding graph is depicted on Figure 2.

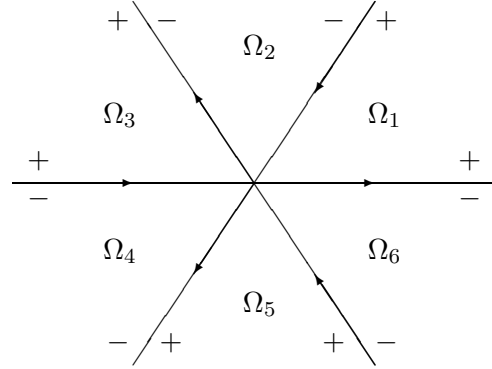


Figure 2: The oriented contour Σ .

The jump matrix has the form:

$$J(k, x, t) = \begin{cases} \begin{pmatrix} 1 & \lambda \bar{r}(k) e^{-2i\theta} \\ -r(k) e^{2i\theta} & 1 - \lambda |r(k)|^2 \end{pmatrix} & \arg k = 0, \pi; \\ \begin{pmatrix} 1 & 0 \\ c(k) e^{2i\theta} & 1 \end{pmatrix} & \arg k = \frac{\pi}{3}, \frac{2\pi}{3}; \\ \begin{pmatrix} 1 & \lambda \bar{c}(\bar{k}) e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & \arg k = \frac{4\pi}{3}, \frac{5\pi}{3}. \end{cases}$$

The proof of equations (5.5) is a simple algebraic verification of relations (5.1)-(5.4).

Remark. The above Riemann-Hilbert problem is written for the case when the set of the eigenvalues is empty. More details about Riemann-Hilbert problem for the mKdV equation and complete consideration of the initial-boundary value problem can be found in [11].

Remark. We consider the problem for the form (1.1) of the mKdV equation in the first quarter ($x \geq 0, t \geq 0$) of the xt -plane. This problem is different from that studied in [11], which is the initial boundary value problem for the same equation, but in the second quarter ($x \leq 0, t \geq 0$) of the xt -plane. Scattering (spectral) data for that problem have different analytic properties. It is well-known that for the KdV and mKdV equations there are differences between the initial boundary value problems for $x > 0$ and for $x < 0$.

Indeed, the kernel of the Marchenko integral equations, which have to considered now in the domain $-\infty < y < x \leq 0$, takes the form :

$$\begin{aligned} H(x, t) = & \frac{1-\lambda}{2} \left(\sum_{k_j \in \Omega_5} \hat{m}_j^5 e^{ik_j x + 8ik_j^3 t} + \sum_{z_j \in \Omega_4} m_j^4 e^{iz_j x + 8iz_j^3 t} + \sum_{z_j \in \Omega_6} m_j^6 e^{iz_j x + 8iz_j^3 t} \right) \\ & + \frac{1}{2\pi} \int_{\partial\Omega_5} c(k) e^{ikx + 8ik^3 t} dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} (r(k) + c(k)) e^{ikx + 8ik^3 t} dk, \end{aligned}$$

where $c(k)$ is now meromorphic (analytic for $\lambda = 1$) in $\Omega_4 \cup \Omega_6$. For $t = 0$ it is easy to prove that

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial\Omega_5} c(k) e^{ikx} dk = & -\frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dk \\ & + \frac{1-\lambda}{2} \left(- \sum_{z_j \in \Omega_4} m_j^4 e^{iz_j x} - \sum_{z_j \in \Omega_6} m_j^6 e^{iz_j x} + \sum_{k_j \in \Omega_4} \hat{m}_j^4 e^{ik_j x} + \sum_{k_j \in \Omega_6} \hat{m}_j^6 e^{ik_j x} \right) \end{aligned}$$

by using analytical properties of function $c(k)$ in the domain $\Omega_4 \cup \Omega_6$ and corresponding residues. Finally we obtain

$$H(x, 0) = \frac{1-\lambda}{2} \left(\sum_{k_j \in \mathbb{C}_-} \hat{m}_j e^{ik_j x} \right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk.$$

Marchenko equations with this kernel correspond precisely to the initial function $u(x)$.

To prove that $q(x, t)$ satisfies the boundary condition we have also to formulate the Riemann-Hilbert problem for the t -equation. To do so let us define the sectionally meromorphic (analytic for $\lambda = 1$) matrix $N(k, t)$:

$$N(k, t) = \begin{cases} \begin{pmatrix} \frac{\hat{\Psi}_1^+(t, k)e^{-4ik^3t}}{p_2^+(k)} & \hat{\varphi}_1^-(t, k)e^{4ik^3t} \\ \frac{\hat{\Psi}_2^+(t, k)e^{-4ik^3t}}{p_2^+(k)} & \hat{\varphi}_2^-(t, k)e^{4ik^3t} \end{pmatrix} & k \in \Omega_1 \cup \Omega_3 \cup \Omega_5 \\ \begin{pmatrix} \hat{\varphi}_1^+(t, k)e^{-4ik^3t} & \frac{\hat{\Psi}_1^-(t, k)e^{4ik^3t}}{p_1^-(k)} \\ \hat{\varphi}_2^+(t, k)e^{-4ik^3t} & \frac{\hat{\Psi}_2^-(t, k)e^{4ik^3t}}{p_1^-(k)} \end{pmatrix} & k \in \Omega_2 \cup \Omega_4 \cup \Omega_6, \end{cases}$$

where $\hat{\varphi}(t, k) = (\hat{\varphi}^-(t, k) \ \hat{\varphi}^+(t, k))$ and $\hat{\Psi}(t, k) = (\hat{\Psi}^-(t, k) \ \hat{\Psi}^+(t, k))$ are matrix solutions (2.9) and (2.10) of the t -equation. Then it is easy to verify that $N(k, t)$ is a solution of the following Riemann-Hilbert problem:

$$N_-(k, t) = N_+(k, t)J^t(k, t), \quad k \in \Sigma \quad (5.6)$$

on the contour Σ (Figure 2) oriented as above. The jump matrix $J^t(k, t)$ has the form:

$$J^t(k, t) = \begin{pmatrix} 1 & p_-(k)e^{4ik^3t} \\ -p_+(k)e^{-4ik^3t} & 1 - p_-(k)p_+(k) \end{pmatrix},$$

where $p_-(k) = p_2^-(k)/p_1^-(k)$, $p_+(k) = p_1^+(k)/p_2^+(k)$, and $p_j^\pm(k)$ are the entries of the “scattering” matrix $P(k)$.

We have already proved that $q(x, 0) = u(x)$. The proof that $q(x, t)$ satisfies the boundary values $q(0, t) = v(t)$, $q'_x(0, t) = v_1(t)$ and $q''_{xx}(0, t) = v_2(t)$ is carried out by using Riemann-Hilbert problems (5.5) and (5.6) in the same manner as in [28]. The main tool in the proof is the existence of an analytic map from the Riemann-Hilbert problem (5.5) attached to $M(k, 0, t)$ into the Riemann-Hilbert problem (5.6) attached to $N(k, t)$. Such a proof for the mKdV equation is precisely given in [11].

6 Inverse scattering Problem

Let \mathcal{R} be scattering data (4.10) satisfying conditions A–B–C. Then:

Statements. 1. *The xt -integral equation*

$$K(x, y, t) + \mathcal{H}(x + y, t) + \int_x^\infty K(x, z, t)\mathcal{H}(z + y, t)dz = 0, \quad (6.1)$$

$$0 \leq x < y < \infty, \quad 0 \leq t < \infty$$

with the 2×2 matrix kernel

$$\mathcal{H} = \begin{pmatrix} 0 & H(x, t) \\ \lambda H(x, t) & 0 \end{pmatrix},$$

where real scalar function $H(x, t)$ given by (4.8), is uniquely solvable in $L^1(x, \infty)$ for any $x \geq 0$ and $t \geq 0$.

2. The solution $K(x, y, t)$ belongs to $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$, it and all its derivatives decrease faster than any negative power of $x + y$, for $x + y \rightarrow \infty$, and t fixed.

3. The matrix

$$\Psi(x, t, k) = \left[e^{-ikx\sigma_3} + \int_x^\infty K(x, y, t) e^{-iky\sigma_3} dy \right] e^{-4ik^3 t \sigma_3}$$

satisfies the symmetry conditions

$$\begin{aligned} \Psi(x, t, k) &= \Lambda \bar{\Psi}(x, t, k) \Lambda^{-1} \quad \text{for } k \in \mathbb{R} \\ \Psi^\pm(x, t, k) &= \overline{\Psi^\pm(x, t, -\bar{k})} \quad \text{for } k \in \mathbb{C}_\pm \end{aligned}$$

and is a solution of the x -equation (1.2) with $Q(x, t)$ given by

$$Q(x, t) = \sigma_3 K(x, x, t) \sigma_3 - K(x, x, t). \quad (6.2)$$

4. $\Psi(x, t, k)$ is a solution of the x - and t -equations constructed from the matrix $Q(x, t)$ and its derivative $Q'_x(x, t)$, $Q''_{xx}(x, t)$, using eqs (6.2), (1.2), (1.3) and (2.3).

5. The scattering data \mathcal{R} of these compatible differential equations coincide with the chosen function $r(k)$, the function $c(k)$ and the numbers $k_1, k_2, \dots, k_n \in \mathbb{C}_+$, $z_1, z_2, \dots, z_m \in \Omega_2$.

Statement 1 follows from Lemma 2 about the solvability of the xt -integral equations:

Lemma 2. Let \mathcal{R} be scattering data satisfying conditions A–B–C. Then the xt -integral equations (4.6)–(4.8) have a unique solution in $L^1(x, \infty)$.

Proof. Under conditions A–B–C the integral operator of the xt -integral equation is compact in $L^1(x, \infty)$. Then, by Fredholm theory the xt -integral equation has a unique solution if the homogeneous equation has no non-zero solution. If a non-zero solution does exist in $L^1(x, \infty)$, in view of the homogeneity of the integral equation, it is bounded, hence belongs to $L^2(x, \infty)$. The integral operator is clearly skew-Hermitian in $L^2(x, \infty)$, so we obtain a contradiction, because the only solution in this case is zero. For $\lambda = 1$ the proof is more complicated. For example, it follows from the solvability of the corresponding Riemann-Hilbert problem (5.5). In turn, the unique solvability of the Riemann-Hilbert problem is proved by the same way as in [50]. \square

Statement 2 follows from Lemma 3:

Lemma 3. *Let conditions A–B–C be fulfilled and $(K_1(x, y, t), K_2(x, y, t))$ be the solution of the xt -integral equations (4.6)–(4.8).*

Then $(K_1(x, y, t), K_2(x, y, t)) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$. These functions and all their derivatives decrease faster than any negative power of $x + y$, for $x + y \rightarrow \infty$, t fixed. Moreover,

$$q(x, t) = -2\lambda K_2(x, x, t)$$

- (a) *is C^∞ in x and t ,*
- (b) *decreases faster than any negative power of x for $x \rightarrow \infty$, t fixed,*
- (c) *is a solution of the mKdV equation with initial function $q(x, 0) = u(x)$ and boundary values $q(0, t) = v(t)$, $q'_x(0, t) = v_1(t)$, $q''_{xx}(0, t) = v_2(t)$.*

Proof. According to Lemma 2 the xt -integral equations have a solution

$$K(x, y, t) = \begin{pmatrix} K_1(x, y, t) & \lambda K_2(x, y, t) \\ K_2(x, y, t) & K_1(x, y, t) \end{pmatrix}$$

which belongs to $L^1(x, \infty)$. By condition A, the kernel $H(x+y, t)$ is in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and is fast decreasing as $x + y \rightarrow \infty$ (end of Section 4). Therefore $(K_1(x, y, t), K_2(x, y, t))$ is in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)$ and vanishes faster than any negative power of $x + y$ as $x + y \rightarrow \infty$, t fixed. The same is true for all their derivatives and for $q(x, t)$. It is clear that $H(x, t)$ satisfies:

$$\frac{\partial}{\partial t} H(x, t) + 8 \frac{\partial^3}{\partial x^3} H(x, t) = 0.$$

Then it is well-known that $q(x, t)$ solves the mKdV equation, cf. [1, 49]. The fact that $q(x, t)$ satisfies the boundary conditions was discussed at the end of Section 5. \square

Proof of Statements 3 and 4. The proof of Statement 3 is well-known [1, 19]. In particular, formula (6.2) follows from equation (2.4). Statement 4 is also true. Indeed, due to Lemma 3, the function $q(x, t)$ is a solution of the mKdV equation (1.1). Hence the constructed x - and t -equations are compatible. Therefore, according to Lemma 1, the matrix-valued function $\Psi(x, t, k)$ solves the t -equation. \square

Proof of Statement 5. Let us consider compatible solutions $\Psi(x, t, k)$ (2.3), $\Phi(x, t, k)$ (2.6) and $Y(x, t, k)$ (2.11) of the constructed x - and t -equations. Let $\tilde{\mathcal{R}}$ be the corresponding scattering data. We have to show that $\tilde{\mathcal{R}} \equiv \mathcal{R}$.

First of all one can find that the matrix $K(x, y, 0)$ solves the xt -integral equations ($t = 0$) with kernel $\mathcal{H}(x, 0)$ generated by

$$\tilde{H}(x) = \sum_{\substack{\tilde{k}_j \in \mathbb{C}_+ \\ \tilde{s}_2^+(\tilde{k}_j) = 0}} \tilde{m}_j e^{i\tilde{k}_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{r}(k) e^{ikx} dk.$$

On the other hand the matrix $K(x, y, 0)$ solves the same integral equation with kernel generated by

$$H(x) = \sum_{\substack{k_j \in \mathbb{C}_+ \\ s_2^+(k_j) = 0}} m_j e^{ik_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk,$$

since

$$\frac{1}{2\pi} \int_{\partial\Omega_2} c(k) e^{ikx} dk = \sum_{\substack{k_j \in \Omega_2 \\ s_2^+(k_j)=0}} m_j^2 e^{ik_j x} - \sum_{\substack{z_j \in \Omega_2 \\ r_1^-(z_j)=0}} m_j^2 e^{iz_j x}.$$

Hence $F(x) = \tilde{H}(x) - H(x)$ is a solution of the homogeneous Volterra integral equation

$$F(2x) + \int_{2x}^{\infty} K_1(x, y-x, 0) F(y) dy = 0,$$

which yields the identity $F(x) \equiv 0$. Therefore $\tilde{k}_j = k_j$, $\tilde{m}_j = m_j$ (hence $\tilde{m}_j^1 = m_j^1$, $\tilde{m}_j^3 = m_j^3$) and $\tilde{r}(k) \equiv r(k)$ for $k \in \mathbb{R}$. For $t > 0$ by using (4.8) we shall obtain now the relation $F(x, t) = \tilde{H}(x, t) - H(x, t) \equiv 0$, that yields $\tilde{c}(k) \equiv c(k)$ and $\tilde{z}_j = z_j$, $\tilde{m}_j^2 = m_j^2$. Hence $\tilde{\mathcal{R}} \equiv \mathcal{R}$. \square

All statements on the inverse scattering problem are proved. The main theorem is proved.

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